Random Eigenvalue Problems in Structural Analysis

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A computerized Monte Carlo simulation is presented for calculating the statistical properties of the eigenvalues of a spring supported beam-column. The spring supports and axial force are treated as random variables; the distributions of material and geometric properties are considered to be correlated homogeneous random functions. Each sample distribution is generated using a new method for simulating multivariate homogeneous random processes having a specified cross-spectral density matrix. This method of solution is used to investigate the accuracy of the perturbation method for calculating the variance of the *n*th vibration and buckling eigenvalues. Numerical results are presented for the case where the axial load is equal to 27% of the fundamental buckling load and the distributions of material and geometric properties are uncorrelated. The perturbation method is shown to be acceptable for limited ranges of the statistical variations of properties.

Introduction

MANY engineering materials, including modern fiber reinforced composite materials, concrete, soils etc., exhibit a considerable degree of spacewise randomness in their thermomechanical properties. Similarly, the components of structural and mechanical systems often exhibit considerable statistical variations in their properties, so that characteristics of the structure depending on these properties—such as its eigenvalues—will also show some statistical variation.

It is of considerable practical importance to estimate the statistical properties of random eigenvalues, in particular the expected value and variance. One would like to be able to obtain an exact expression for these quantities as a function of the statistical variations in the properties of the system. In general, however, one has to be content with a formula derived using various approximations. One such method is to assume that the statistical variations in the properties are small, thereby allowing the use of perturbation methods. In particular, a number of papers[1–6] have been published in recent years where perturbation techniques were used to derive expressions for the expected value and variance of the nth vibration and buckling eigenvalues. In one instance results were given for the case of large statistical variations in the properties, which seemed to violate the assumption inherent in perturbation methods, namely that the variations be small. This raised the question of just how large the statistical variations can be before the perturbation method breaks down. In this study, a Monte Carlo simulation is used to obtain accurate estimates of the variance of the nth vibration and buckling eigenvalues of a beam-column with random properties. Further, the results obtained using the Monte Carlo simulation are compared with the corresponding results obtained using the perturbation method of Ref. 1, extended to include buckling eigenvalues, thereby giving the ranges of the statistical variations of properties over which the perturbation method yields acceptable estimates for the variance of the eigenvalues.

The beam-column considered (see Fig. 1) is supported at its ends by rotary springs. The spring supports and axial force are treated as random variables; the distributions of material and geometric properties are considered to be correlated homogeneous random functions. The distributions of these properties are generated using a new method[7] for simulating multivariate homogeneous random processes having a specified cross-spectral density matrix.

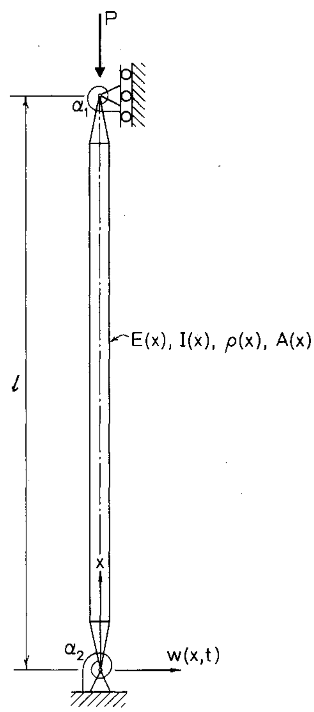

Fig. 1 Beam-column.

The simulation method used consists of generating on a digital computer a random sample of beam-columns and computing the eigenvalues of each beam-column in the sample. The sample mean and variance of these eigenvalues yields an approximation to the exact solution in the following sense. The sample statistic, whether the expected value or the variance, is actually a random variable whose expected value is identical to the exact solution and whose standard deviation approaches zero as the sample size is made infinitely large. Thus, the sample mean and variance are scattered about the exact solution, the amount of scatter decreasing with increasing sample size.

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Assumptions and Basic Equations

Considering the beam-column shown in Fig. 1, neglecting the effect of shear and rotary inertia and making the usual assumptions,[1] one obtains the following governing differential equation and boundary conditions:

$$\partial^2/\partial x^2[E(x)I(x)\partial^2 w(x,t)/\partial x^2] + P\,\partial^2 w(x,t)/\partial x^2 + \rho(x)A(x)\partial^2 w(x,t)/\partial t^2 = 0 \quad (1)$$

$$E(0)I(0)\partial^2 w(0,t)/\partial x^2 - \alpha_1\,\partial w(0,t)/\partial x = 0;\quad w(0,t) = 0 \quad (2)$$

$$E(l)I(l)\partial^2 w(l,t)/\partial x^2 + \alpha_2\,\partial w(l,t)/\partial x = 0;\quad w(l,t) = 0 \quad (3)$$

where $\alpha_1 = \alpha_2 = 0$ for simple supports and $\alpha_1 = \alpha_2 = \infty$ for fixed supports. In the vibration case, the motion is assumed to be simple harmonic with circular frequency ω, so that the term $\partial^2 w/\partial t^2$ is replaced by $-\omega^2 w$. Furthermore, with the following substitutions,

$$x = \xi l \quad (4)$$

$$E(\xi) = E_0[1 + a(\xi)];\quad I(\xi) = I_0[1 + b(\xi)] \quad (5)$$

$$\rho(\xi) = \rho_0[1 + c(\xi)];\quad A(\xi) = A_0[1 + d(\xi)] \quad (6)$$

$$\alpha_1 = \alpha_1^0(1 + s);\quad \alpha_2 = \alpha_2^0(1 + u) \quad (7)$$

$$\lambda = \rho_0 A_0 l^4 \omega^2/E_0 I_0;\quad \mu = Pl^2/E_0 I_0 \quad (8)$$

$$R(\xi) = a(\xi) + b(\xi) + a(\xi)b(\xi);\quad S(\xi) = c(\xi) + d(\xi) + c(\xi)d(\xi) \quad (9)$$

the differential equation and boundary conditions reduce to,

$$\{[1 + R(\xi)]w''(\xi)\}'' + \mu w''(\xi) - \lambda[1 + S(\xi)]w(\xi) = 0 \quad (10)$$

$$[1 + R(0)]w''(0) - (\alpha_1^0 l/E_0 I_0)(1 + s)w'(0) = 0;\quad w(0) = 0 \quad (11)$$

$$[1 + R(1)]w''(1) + (\alpha_2^0 l/E_0 I_0)(1 + u)w'(1) = 0;\quad w(1) = 0 \quad (12)$$

where a prime denotes differentiation with respect to ξ. In the static buckling case, μ is unknown and $\lambda = 0$. In the vibration case μ is known and further substitutions are introduced;

$$P = P_0(1 + v);\quad \mu_0 = P_0 l^2/E_0 I_0;\quad \mu = \mu_0(1 + v) \quad (13)$$

so that Eq. (10) is replaced by

$$\{[1 + R(\xi)]w''(\xi)\}'' + \mu_0(1 + v)w''(\xi) - \lambda[1 + S(\xi)]w(\xi) = 0$$

in which λ is the unknown. Equations (10–12) describe an eigenvalue problem. It has nontrivial solutions $w(\xi) = X_n(\xi)$ corresponding to the eigenvalues $\lambda = \lambda_n$ (vibration) and $\mu = \mu_n$ (buckling) where $n = 1, 2, \ldots$. The eigenfunction $X_n(\xi)$ corresponds to either the nth vibration mode shape or the nth buckling mode shape. The frequency of the nth vibration mode is given by $\omega_n^2 = \lambda_n E_0 I_0/\rho_0 A_0 l^4$, whereas the nth buckling load is given by $P_n = \mu_n E_0 I_0/l^2$.

The assumptions regarding the statistical properties of the beam-column can be restated as: 1) The nondimensional spring constants s and u are random variables with zero mean; 2) The nondimensional axial load v is a time-independent random variable with zero mean; 3) The nondimensional distributions of Young's modulus $a(\xi)$ and material density $c(\xi)$ are correlated homogeneous random functions with zero mean; 4) The nondimensional distributions of cross-sectional area $d(\xi)$ and areal moment of inertia $b(\xi)$ are correlated homogeneous random functions with zero mean; 5) Apart from the correlation just referred to, all random functions and random variables are considered to be statistically independent.

Perturbation Method

In applying the perturbation method, s, u, v, $a(\xi)$, $b(\xi)$, $c(\xi)$, and $d(\xi)$ are considered to be small perturbations about an unperturbed case ($s = u = v = a(\xi) = b(\xi) = c(\xi) = d(\xi) = 0$). These perturbations are considered to be small if their magnitudes are small compared to unity. This assumption can be relaxed somewhat once the ensemble average of the eigenvalues has been obtained.

Referring the reader to Ref. 1 for details regarding the vibration case (the buckling case closely parallels the vibration case), the result of the standard perturbation analysis for λ_n and μ_n is listed below.

$$\lambda_n = \lambda_n^0 + (1/D_n)\int_0^1 R(\xi)I_n(\xi)d\xi - (\lambda_n^0/D_n)\int_0^1 S(\xi)F_n(\xi)d\xi + (\alpha_1^0 l/E_0 I_0)[H_n(0)/D_n]s + (\alpha_2^0 l/E_0 I_0)[H_n(1)/D_n]u + \mu_0(E_n/D_n)v \quad (14)$$

$$\mu_n = \mu_n^0 + (1/D_n)\int_0^1 R(\xi)I_n(\xi)d\xi + (\alpha_1^0 l/E_0 I_0)[H_n(0)/D_n]s + (\alpha_2^0 l/E_0 I_0)[H_n(1)/D_n]u \quad (15)$$

where,

$$F_n(\xi) = [X_n^0(\xi)]^2;\quad H_n(\xi) = [X_n^{0\prime}(\xi)]^2;\quad I_n(\xi) = [X_n^{0\prime\prime}(\xi)]^2;$$

$$E_n = \int_0^1 X_n^{0\prime\prime}(\xi)X_n^0(\xi)d\xi \quad (16)$$

$$D_n = \begin{cases} \displaystyle\int_0^1 [X_n^0(\xi)]^2\,d\xi & \text{vibration} \\ \displaystyle\int_0^1 [X_n^{0\prime}(\xi)]^2\,d\xi & \text{buckling} \end{cases} \quad (17)$$

and λ_n^0, μ_n^0, $X_n^0(\xi)$ refer to the unperturbed case. Although Eqs. (14) and (15) were obtained by a rather lengthy perturbation analysis, it is interesting to observe that these equations can be derived very quickly by first obtaining the Rayleigh quotient for λ_n and μ_n from Eqs. (10–12), then using, as the assumed mode shape, the mode shape $X_n^0(\xi)$ of the unperturbed case. The fact that the perturbation analysis yields a Rayleigh quotient approximation to λ_n and μ_n may account for the fact, as shown later, that the perturbation method predicts the expected value and variance of λ_n and μ_n with reasonable accuracy for rather large perturbations.

Equations (14) and (15) are now used to derive the expected value and variance of λ_n and μ_n. In effect, these equations are used to calculate λ_n and μ_n for each sample beam-column to find the average and variance. In so doing, the previous requirement that every realization have small perturbations is somewhat relaxed to that requiring the standard deviations of the perturbations to be small.

Taking the expected value of Eqs. (14) and (15), and using the assumptions made with regard to statistical independence of the random variables and random functions, one obtains

$$E[\lambda_n] = \lambda_n^0;\quad E[\mu_n] = \mu_n^0 \quad (18)$$

which are independent of any random perturbations and equal to the corresponding eigenvalues for the unperturbed case. Squaring Eqs. (14) and (15) and repeating the aforementioned procedure leads to the following expressions for the variance of λ_n and μ_n:

$$\begin{aligned}\mathrm{Var}[\lambda_n] = {} & (1/D_n^2)\int_0^1\int_0^1 E[R(\xi_1)R(\xi_2)]I_n(\xi_1)I_n(\xi_2)d\xi_1\,d\xi_2 \\ & - (2\lambda_n^0/D_n^2)\int_0^1\int_0^1 E[R(\xi_1)S(\xi_2)]I_n(\xi_1)F_n(\xi_2)d\xi_1\,d\xi_2 \\ & + (\lambda_n^{02}/D_n^2)\int_0^1\int_0^1 E[S(\xi_1)S(\xi_2)]F_n(\xi_1)F_n(\xi_2)d\xi_1\,d\xi_2 \\ & + \{(\alpha_1^0 l/E_0 I_0)[H_n(0)/D_n]\}^2\sigma_s^2 \\ & + \{(\alpha_2^0 l/E_0 I_0)[H_n(1)/D_n]\}^2\sigma_u^2 + [\mu_0 E_n/D_n]^2\sigma_v^2 \end{aligned} \quad (19)$$

$$\begin{aligned}\mathrm{Var}[\mu_n] = {} & (1/D_n^2)\int_0^1\int_0^1 E[R(\xi_1)R(\xi_2)]I_n(\xi_1)I_n(\xi_2)d\xi_1\,d\xi_2 \\ & + \{(\alpha_1^0 l/E_0 I_0)[H_n(0)/D_n]\}^2\sigma_s^2 \\ & + \{(\alpha_2^0 l/E_0 I_0)[H_n(1)/D_n]\}^2\sigma_u^2 \end{aligned} \quad (20)$$

where σ_s^2, σ_u^2, and σ_v^2 are variances of s, u, and v, respectively. Substituting from Eq. (9)

$$E[R(\xi_1)R(\xi_2)] = R_a(\xi_1 - \xi_2) + R_b(\xi_1 - \xi_2) + R_a(\xi_1 - \xi_2)R_b(\xi_1 - \xi_2)$$

$$E[R(\xi_1)S(\xi_2)] = R_{ac}(\xi_1 - \xi_2) + R_{bd}(\xi_1 - \xi_2) + R_{ac}(\xi_1 - \xi_2)R_{bd}(\xi_1 - \xi_2)$$

$$E[S(\xi_1)S(\xi_2)] = R_c(\xi_1 - \xi_2) + R_d(\xi_1 - \xi_2) + R_c(\xi_1 - \xi_2)R_d(\xi_1 - \xi_2)$$

where, for example, $R_a(\xi_1 - \xi_2)$ is the autocorrelation function of $a(\xi_1)$ and $a(\xi_2)$, and $R_{ac}(\xi_1 - \xi_2)$ is the cross-correlation function of $a(\xi_1)$ and $c(\xi_2)$.

Equations (18–20) indicate that the perturbation method requires a knowledge of the nth eigenvalue and the nth eigenfunction for the unperturbed case, that is, a uniform beam-column with spring constants α_1^0 and α_2^0 and subjected to an axial load P_0. Dropping the superscript or subscript 0, the vibration eigenvalues for the uniform beam-column are obtained as solutions to the transcendental equation in λ:

$$\begin{aligned} &\sin k_1 \sinh k_2/k_1k_2 + \Lambda[(\sin k_1 \cosh k_2/k_1) \\ &\quad - (\cos k_1 \sinh k_2/k_2)][(\alpha_1 l/EI) + (\alpha_2 l/EI)] \\ &\quad + 2\Lambda^2(1 - \cos k_1 \cosh k_2 \\ &\quad - \mu \sin k_1 \sinh k_2/2k_1k_2)(\alpha_1 l/EI)(\alpha_2 l/EI) = 0 \end{aligned} \tag{21}$$

where,

$$k_1, k_2 = \{[(\mu/2)^2 + \lambda]^{1/2} \pm \mu/2\}^{1/2}; \quad \Lambda = 1/(k_1^2 + k_2^2)$$

Adding a subscript n to refer to the nth eigenvalue, the nth eigenfunction is then given by

$$X_n(\xi) = C_{1n} \cos k_{1n}\xi + C_{2n} \sin k_{1n}\xi + C_{3n} \cosh k_{2n}\xi + C_{4n} \sinh k_{2n}\xi \tag{22}$$

The buckling eigenvalues are obtained as solutions to the transcendental equation in μ:

$$\begin{aligned} &2(\alpha_1 l/EI)(\alpha_2 l/EI) \\ &\quad - \{2(\alpha_1 l/EI)(\alpha_2 l/EI) + k^2[(\alpha_1 l/EI) + (\alpha_2 l/EI)]\}\cos k \\ &\quad + \{k^2 - (\alpha_1 l/EI)(\alpha_2 l/EI) \\ &\quad + [(\alpha_1 l/EI) + (\alpha_2 l/EI)]\}k \sin k = 0 \end{aligned} \tag{23}$$

where $k = \mu^{1/2}$. The nth eigenfunction is then given by

$$X_n(\xi) = C_{1n} \cos k_n\xi + C_{2n} \sin k_n\xi + (C_{3n}\xi + C_{4n})/k_n^2 \tag{24}$$

Equations (21) and (23) can be solved by a trial and error procedure and shown to reduce to the standard transcendental equations for the case when the beam-column is pinned-pinned, pinned–fixed, and fixed-fixed. Hoshiya and Shah[1] did not obtain the exact solution for the unperturbed case when the beam-column is not simply supported. They considered only the fundamental vibration mode, approximating it by the polynomial;

$$X_1^0(\xi) = \xi + a_1\xi^2 + a_2\xi^3 + a_3\xi^4 \tag{25}$$

in which a_1, a_2, a_3 are chosen so that the spring support boundary conditions are satisfied. The fundamental vibration eigenvalue was then approximated by the Rayleigh quotient using Eq. (25) as the assumed mode shape. This procedure can also be used for the buckling case. The use of both the exact and the assumed mode shapes to evaluate Eqs. (19) and (20) is considered in the present study.

Simulation Method

The simulation method consists of generating a sample of beam-columns and computing their eigenvalues. For each realization of the beam-column, one has to generate 1) random spring constants α_1 and α_2 and end load P, and 2) the spatial distributions of E, I, ρ, and A. After thus obtaining a sample for each of the eigenvalues λ_i and μ_i ($i = 1, 2, \ldots, n$), the sample mean and variance is taken as an approximation to the corresponding mean and variance of the parent population.

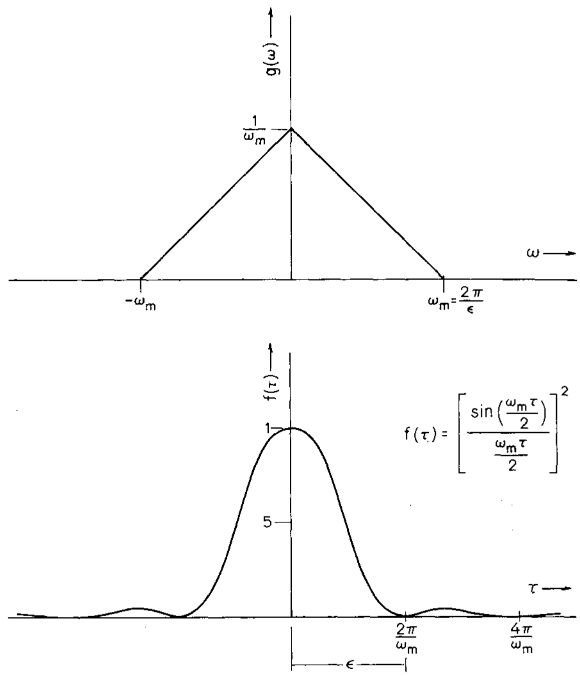

Fig. 2 Functions $g(\omega)$ and $f(\tau)$.

Generating α_1, α_2 and P

$\mathrm{Var}[\lambda_n]$ and $\mathrm{Var}[\mu_n]$, as calculated by the perturbation method, depend only on the variances of s, u, and v. It can therefore be inferred that the shape of the probability distribution curve affects only the higher-order statistics not considered here. Hence, probability distributions for the simulation can be chosen arbitrarily. For convenience, uniform distributions are chosen in the present study; s is uniformly distributed over the interval $[-(3)^{1/2}\sigma_s, (3)^{1/2}\sigma_s]$ and similarly for u and v. For each realization of the random variables s, u, and v, the corresponding values for α_1, α_2, and P are found using Eqs. (7) and (13).

Generating $E(\xi)$, $I(\xi)$, $\rho(\xi)$ and $A(\xi)$

A particular case is considered where the random functions $a(\xi), \ldots, d(\xi)$ have the following cross-spectral density matrices:

$$S_{ac}(\omega) = \begin{bmatrix} \sigma_a^2 & \rho_{ac}\sigma_a\sigma_c \\ \rho_{ac}\sigma_a\sigma_c & \sigma_c^2 \end{bmatrix} g(\omega)$$

$$S_{bd}(\omega) = \begin{bmatrix} \sigma_b^2 & \rho_{bd}\sigma_b\sigma_d \\ \rho_{bd}\sigma_b\sigma_d & \sigma_d^2 \end{bmatrix} g(\omega) \tag{26}$$

where the function g is shown in Fig. 2 with a maximum wave number content ω_m, and $\rho_{ac}^2(\rho_{bd}^2)$ is the coherence function determining the cross-correlation between Young's Modulus and material density (between areal moment of inertia and cross-sectional area). The wave length $\varepsilon = 2\pi/\omega_m$ associated with ω_m can serve as correlation distance, a distance along the beam-column over which appreciable correlation occurs within each random function. Thus ε is a measure of how rapidly, for example, the material and geometric properties can vary along the beam-column; the smaller the value of ε, the more rapidly the properties can change from point to point.

Applying the Weiner-Khintchine relations to Eq. (26), one finds that the random functions $a(\xi), \ldots, d(\xi)$ have cross-correlation matrices of the form of Eq. (26) with $g(\omega)$ replaced by its Weiner-Khintchine transform $f(\tau)$ (see Fig. 2). Using these expressions for the auto and cross-correlation functions, we find that the perturbation solution, as given by Eqs. (19) and (20), reduces to:

$$\mathrm{Var}[\lambda_n] = (1/D_n^2)\begin{bmatrix}(\sigma_a^2+\sigma_b^2)I_{21} - 2\lambda_n^0(\sigma_{ac}^2+\sigma_{bd}^2)I_{22} \\ +\lambda_n^{02}(\sigma_c^2+\sigma_d^2)I_{23} + \sigma_a^2\sigma_b^2 I_{41} \\ -2\lambda_n^0\sigma_{ac}^2\sigma_{bd}^2 I_{42} + \lambda_n^{02}\sigma_c^2\sigma_d^2 I_{43} \\ +[(\alpha_1^0 l/E_0I_0)H_n(0)]^2\sigma_s^2 \\ +[(\alpha_2^0 l/E_0I_0)H_n(1)]^2\sigma_u^2 + [\mu_0 E_n]^2\sigma_v^2\end{bmatrix} \tag{27}$$

$$\mathrm{Var}[\mu_n] = (1/D_n^2)\begin{bmatrix}(\sigma_a^2+\sigma_b^2)I_{21} + \sigma_a^2\sigma_b^2 I_{41} \\ +[(\alpha_1^0 l/E_0I_0)H_n(0)]^2\sigma_s^2 \\ +[(\alpha_2^0 l/E_0I_0)H_n(1)]^2\sigma_u^2\end{bmatrix} \tag{28}$$

where,

$$\sigma_{ac}^2 = \rho_{ac}\sigma_a\sigma_c;\quad \sigma_{bd}^2 = \rho_{bd}\sigma_b\sigma_d$$

$$I_{j1} = \int_0^1\int_0^1 S_j(\xi_1-\xi_2)I_n(\xi_1)I_n(\xi_2)d\xi_1\,d\xi_2;\quad j=2,4$$

$$I_{j2} = \int_0^1\int_0^1 S_j(\xi_1-\xi_2)I_n(\xi_1)F_n(\xi_2)d\xi_1\,d\xi_2;\quad j=2,4$$

$$I_{j3} = \int_0^1\int_0^1 S_j(\xi_1-\xi_2)F_n(\xi_1)F_n(\xi_2)d\xi_1\,d\xi_2;\quad j=2,4$$

$$S_j(\xi) = [\sin(\omega_m\xi/2)/(\omega_m\xi/2)]^j;\qquad j=2,4$$

The double integrals are evaluated numerically using Simpson's rule.

Each sample distribution for the random functions E, I, ρ, and A is generated using a method[7] for simulating multivariate stationary random processes having a specified cross-spectral density matrix. A very desirable feature of this method of simulation is its ease of computation. This method permits one to simulate a set of two stationary random processes $f_1(\xi)$ and $f_2(\xi)$, with zero mean and having a specified target cross-spectral density matrix $S_f^0(\omega)$ as follows:

$$f_1(\xi) = \sigma_1(2/N)^{1/2}\sum_{k=1}^{N}\cos\{\omega_{1k}\xi + \phi_{1k}\} \tag{29}$$

$$f_2(\xi) = \sigma_2(2/N)^{1/2}\left[\rho_{12}\sum_{k=1}^{N}\cos\{\omega_{1k}\xi+\phi_{1k}\} + (1-\rho_{12}^2)^{1/2}\sum_{k=1}^{N}\cos\{\omega_{2k}\xi+\phi_{2k}\}\right] \tag{30}$$

where, ω_{jk} $(j = 1, 2; k = 1, 2, \ldots, N)$ = random variables identically and independently distributed with the density function $g(\omega)$; ϕ_{jk} $(j = 1, 2; k = 1, 2, \ldots, N)$ = random variables identically and independently distributed with the uniform density $1/2\pi$ between 0 and 2π. The random functions f_1 and f_2 as defined above are asymptotically ergodic Gaussian processes as $N \to \infty$. In practice, however, a sufficiently large value of N is used to make each process effectively ergodic and Gaussian.

The aforementioned procedure for generating random functions is applied first with $f_1 = a(\xi)$, $f_2 = c(\xi)$, and

$$S_f^0(\omega) = (1/2\pi)\int_{-\infty}^{\infty}\begin{bmatrix}R_a(\tau) & R_{ac}(\tau)\\ R_{ac}(\tau) & R_c(\tau)\end{bmatrix}e^{-i\omega\tau}\,d\tau$$

then with $f_1 = b(\xi)$, $f_2 = d(\xi)$, and

$$S_f^0(\omega) = (1/2\pi)\int_{-\infty}^{\infty}\begin{bmatrix}R_b(\tau) & R_{bd}(\tau)\\ R_{bd}(\tau) & R_d(\tau)\end{bmatrix}e^{-i\omega\tau}\,d\tau$$

thereby generating with the aid of Eqs. (5) and (6) the distributions of E, I, ρ, and A for the particular sample beam-column under consideration.

Calculating λ_i and μ_i $(i = 1, 2, \ldots, n)$

The first four eigenvalues of the beam-column can be calculated accurately by the transfer matrix method[8] by dividing the beam-column into a sufficient number of segments, the number depending upon the wave number ω_m; the larger the value of ω_m the more rapidly the properties can change from point to point and hence the greater the number of segments required.

The preceding procedure is, however, time consuming. Therefore when the beam-column is uniform, the eigenvalues are obtained from Eqs. (21) and (23). In this case, the fundamental eigenvalues are also calculated using the Rayleigh quotient approximation with the assumed mode shape being given by Eq. (25).

Uniform Beam-Column

In this case, $a(\xi) = b(\xi) = c(\xi) = d(\xi) = 0$, so that $\sigma_a = \sigma_b = \sigma_c = \sigma_d = \sigma_{ac} = \sigma_{bd} = 0$ in Eqs. (27) and (28) for the perturbation solution.

The mean and variance of λ_i and μ_i $(i = 1, 2, \ldots, n)$ are calculated by four different methods, namely: 1) from Eqs. (27) and (28) based on the perturbation analysis, 2) computing the sample mean and variance of a sample of 1000 realizations of λ_i or μ_i evaluated from Eqs. (21) or (23), respectively, 3) using an approximation to the perturbation analysis using an assumed mode shape for the unperturbed case to evaluate D_n, H_n, and E_n in Eqs. (27) and (28), and finally 4) computing the sample mean and variance of a sample of 1000 realizations of λ_i or μ_i evaluated using the Rayleigh quotient based upon an assumed mode shape. In each case the assumed mode shape is given by Eq. (25). The first two methods use $n = 4$ whereas the last two methods are for $n = 1$ only.

The comparison of the results obtained by these four methods is given in Figs. 3 and 4 for the fundamental vibration and buckling eigenvalues, respectively. In these and the following figures, V_{α_1} is the coefficient of variation of the spring support stiffness α_1, and similarly for V_{α_2}; P_{CR} is the Euler buckling load of the corresponding simply supported beam-column. The ratio $P/P_{CR} = 1$ yields an axial load equal to 27% of the fundamental buckling load of the spring supported beam-column. Note the labelling of the ordinates: $\mathrm{Var}[\lambda_1] \times 10^{-2} = 6$ implies that $\mathrm{Var}[\lambda_1] = 6 \times 10^{+2}$.

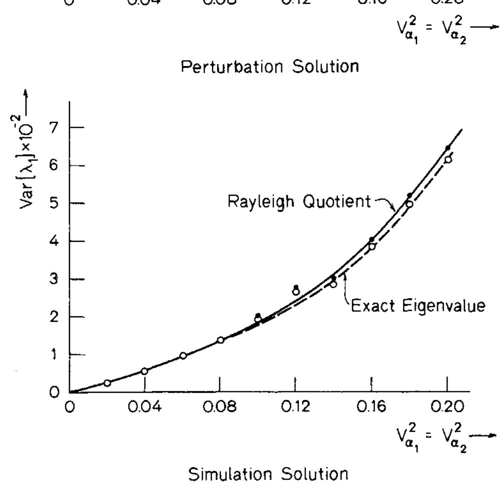

Fig. 3 Variance of fundamental vibration eigenvalue, $\alpha_1^0 l/E_0I_0 = \alpha_2^0 l/E_0I_0 = 50$, $P_0/P_{CR} = 1$, $V_P = 0$.

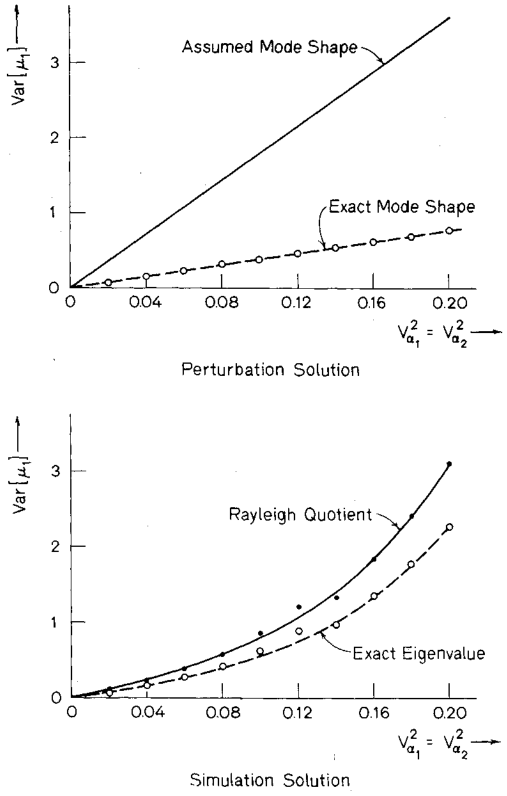

Fig. 4 Variance of fundamental buckling eigenvalue, $\alpha_1^0 l/E_0 I_0 = \alpha_2^0 l/E_0 I_0 = 50$.

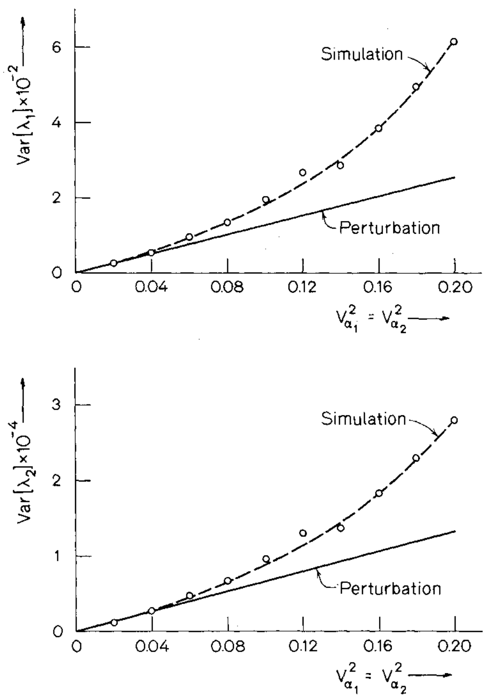

Fig. 5 Variance of first and second vibration eigenvalues, $\alpha_1^0 l/E_0 I_0 = \alpha_2^0 l/E_0 I_0 = 50$, $P_0/P_{CR} = 1$, $V_P = 0$.

Comparison of Fig. 3 with Fig. 4 indicates that in the perturbation method the error resulting from evaluating Eqs. (27) and (28) using an assumed mode shape (a power series satisfying all the boundary conditions) in place of the exact mode shape causes a much greater error in the variance of the fundamental eigenvalue for the buckling case than in the vibration case. In fact, the use of an assumed mode shape to evaluate the perturbation results is unacceptable for the buckling case. Also, it is observed that in the simulation the error resulting from using the Rayleigh quotient in place of the exact eigenvalue causes a greater error in the variance of the fundamental eigenvalue for the buckling case than

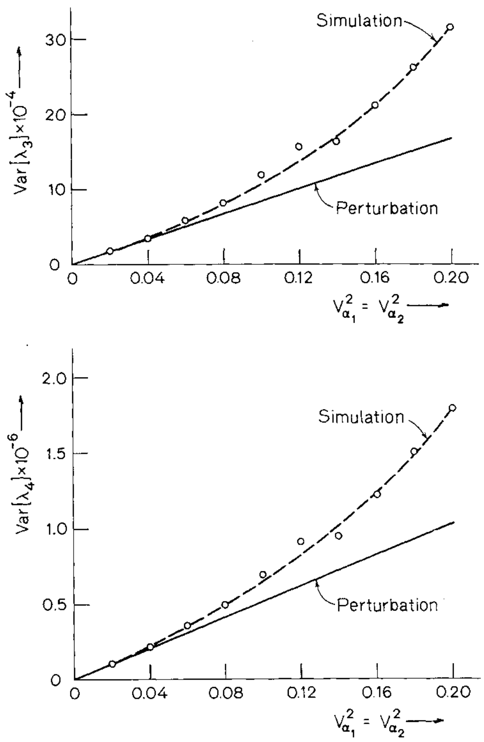

Fig. 6 Variance of third and fourth vibration eigenvalues, $\alpha_1^0 l/E_0 I_0 = \alpha_2^0 l/E_0 I_0 = 50$, $P_0/P_{CR} = 1$, $V_P = 0$.

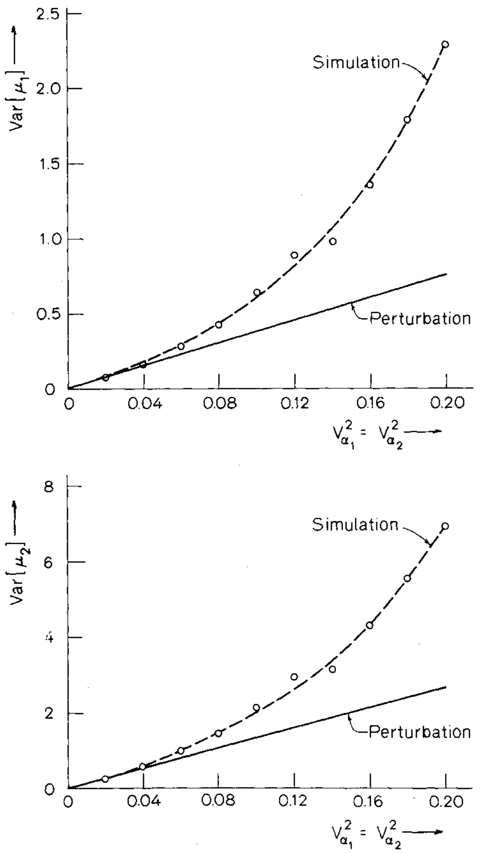

Fig. 7 Variance of first and second buckling eigenvalues, $\alpha_1^0 l/E_0 I_0 = \alpha_2^0 l/E_0 I_0 = 50$.

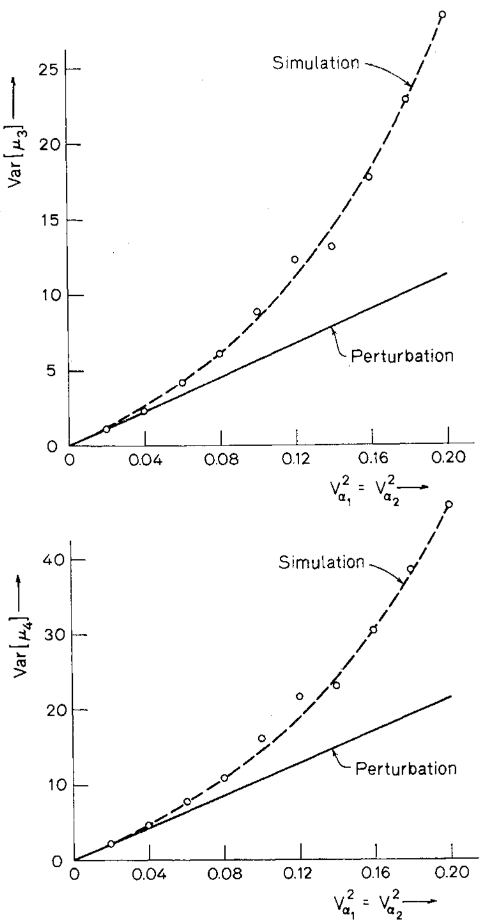

Fig. 8 Variance of third and fourth buckling eigenvalues, $\alpha_1^0 l/E_0 I_0 = \alpha_2^0 l/E_0 I_0 = 50$.

in the vibration case. All the remaining results are based upon an exact evaluation of the eigenvalues and eigenfunctions.

Comparing Figs. 5 and 7, and Figs. 6 and 8, it can be seen that the perturbation method predicts values for the variance of the first four eigenvalues which are less accurate in the buckling case than in the vibration case. However, in either case the perturbation method yields results of sufficient accuracy provided the coefficient of variation of the spring support stiffness is kept within limits. In fact, Figs. 5–8 can be used to determine the limits as a function of the required accuracy. Finally, it is observed in Fig. 9 that the perturbation method can either underestimate or overestimate the variance of the fundamental vibration eigenvalue.

Random Beam-Column

The results shown in Fig. 10 were obtained for the case of fixed end restraints and axial load ($\sigma_s = \sigma_u = \sigma_v = 0$), and with no correlation between the statistical distributions of geometric and material properties ($\rho_{ac} = \rho_{bd} = 0$). Note however, that corresponding results could have been obtained just as easily using nonzero values for these parameters. Further the correlation length was taken equal to the length of the beam-column ($\varepsilon = 1$), the beam was divided into 40 segments, and $N = 500$ in Eqs. (29) and (30). In Fig. 10, the variance of the fundamental vibration and buckling eigenvalues is plotted as a function of the square of the coefficient of variation of Young's Modulus, areal moment of inertia, material density and cross-sectional area. The first two points were obtained using 200 realizations, the next point with 400 realizations and the remaining two points with 600 realizations. The perturbation solution is almost a straight line. Figure 10 indicates that, under the type of correlation considered, the perturbation method provides a reasonable solution for nonuniform beam-columns over a much wider range of statistical variation of E, I, ρ, and A than would be found in reality.

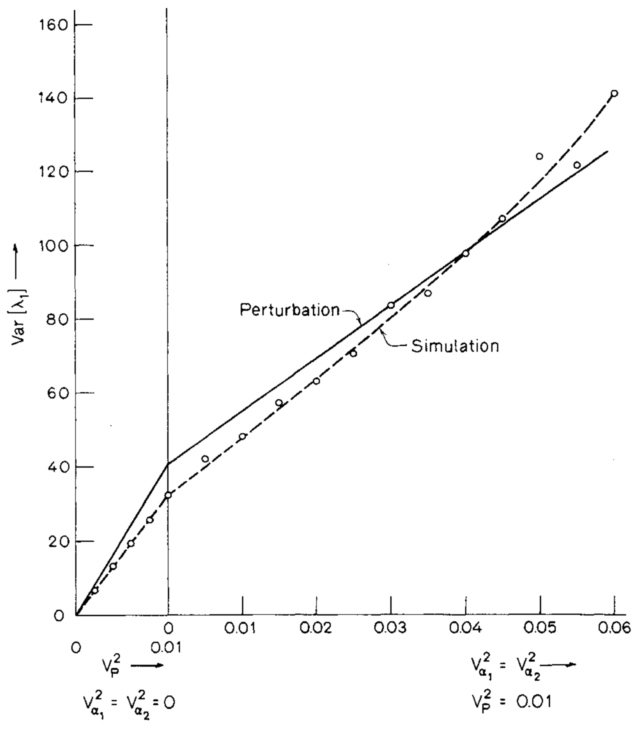

Fig. 9 Variance of fundamental vibration eigenvalue, $\alpha_1^0 l/E_0 I_0 = \alpha_2^0 l/E_0 I_0 = 50$, $P_0/P_{CR} = 0.5$.

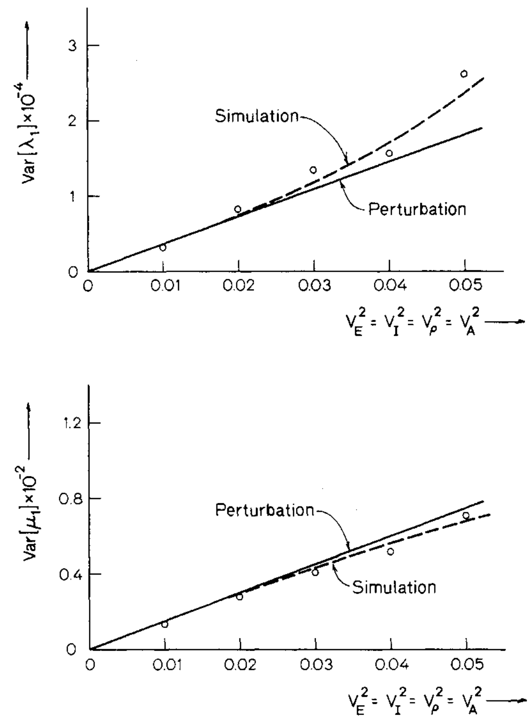

Fig. 10 Variance of fundamental vibration and buckling eigenvalues, $\alpha_1 l/E_0 I_0 = \alpha_2 l/E_0 I_0 = 50$, $P/P_{CR} = 1$.

Summary and Conclusions

A computerized Monte Carlo simulation has been presented for obtaining the expected value and variance of the nth vibration and buckling eigenvalues of a beam-column with random geometric and material properties. Additional sample statistics such as a histogram can also be calculated. The method is versatile, and its extension to more complex structures is limited only by the size of the digital computer and the amount of computing time available.

The method was used to investigate the accuracy of the perturbation method for calculating the variance of the nth vibration and buckling eigenvalues, yielding the following conclusions. For a uniform beam-column with random end restraints: 1) The use of approximate methods, such as the perturbation technique based on an exact or an assumed mode shape, causes a considerably greater error for the buckling case than in the vibration case, 2) The perturbation formula for the variance of the fundamental vibration eigenvalue can be approximated reasonably well by using an assumed mode shape in place of the unperturbed mode shape, whereas the corresponding formula for the buckling case cannot be; the variance calculated using the assumed mode shape can be almost five times the correct result. 3) The perturbation analysis yields results of sufficient accuracy provided the coefficient of variation of the end restraint is kept within reasonable limits. 4) The perturbation analysis for the vibration case can either underestimate or overestimate the variance of the nth eigenvalue.

For a random beam-column with fixed end restraints and axial load, and under the type of correlation considered, the perturbation method provides a reasonable solution over a much wider range of statistical variation of E, I, ρ, and A than would be found in reality.

The aforementioned conclusions were drawn from numerical examples based on the fundamental eigenvalue (random beam-column) or the first four eigenvalues (uniform beam-column) and with an axial load equal to 27% of the fundamental buckling load. It appears that these conclusions will apply also to higher eigenvalues and when the axial load is random.

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